

# Overlap of two Brownian trajectories: exact results for scaling functions

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We consider two random walkers starting at the same time  $t = 0$  from different points in space separated by a given distance  $R$ . We compute the average volume of the space visited by both walkers up to time  $t$  as a function of  $R$  and  $t$  and dimensionality of space  $d$ . For  $d < 4$ , this volume, after proper renormalization, is shown to be expressed through a scaling function of a single variable  $R/\sqrt{t}$ . We provide general integral formulas for scaling functions for arbitrary dimensionality  $d < 4$ . In contrast, we show that no scaling function exists for higher dimensionalities  $d \geq 4$ .

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Statistical properties of the random walk trajectories have been intensively studied for decades. The average volume  $W_1(t)$  visited by a single  $t$ -step Brownian random walk on a  $d$ -dimensional lattice was calculated in the 1960s [1, 2]. This calculation has become a part of extended courses of random walk theory (see, for example [3–5]), and found many applications in reaction-diffusion processes [6] and polymer sciences [7, 8]. A generalization of this classical result to the case of several random walks is of fundamental interest. In the 1990s Larralde and coworkers provided a part of this generalization [9–11]. Namely, they calculated the mean number of sites visited by *at least one* of  $N$  walkers starting from the common origin. Recently, a complementary question was addressed: what is the average number of sites  $W_N(t)$  visited by *all*  $N$  walkers [12]. This quantity as a function of the space dimensionality  $d$  and the number of walkers  $N$  was calculated and its asymptotic behavior for large  $t$  was studied. These results were rederived using the notion of fractal intersections in [14] and further generalized in [13] where the whole distribution of the number of sites visited by  $N$  walkers was calculated exactly for  $d = 1$ .

In this paper, we propose a different generalization of [12]. We consider random walks that, instead of starting altogether from the origin  $x = 0$ , have *distinct* starting points  $x_i$  ( $i = 1 \dots N$ ). The values of  $x_i$  (or, more precisely,  $x_i - x_j$ ) will influence the behavior of  $W_N(t)$ , which now is denoted as  $W_N(t, x_i)$ . For large enough  $t$ , however, random walks “forget” their initial positions, and the position-dependent function  $W_N(t, x_i)$  should converge to  $W_N(t)$  studied in [12]. More generally, one can write

$$W_N(t, x_i) = W_N(t) \Phi_d(\xi_i, \dots, \xi_{N-1}), \quad (1)$$

where  $\Phi_d(\xi_i, \dots, \xi_{N-1})$  is a function of scaling variables  $\xi_i \sim (x_i - N^{-1} \sum_{j=1}^N x_j) / \sqrt{t}$  (exact prefactor will be chosen below) [20], and  $d$  is the dimensionality of space. This scaling function should converge to unity as  $\xi_i \rightarrow 0$  ( $i = 1 \dots N - 1$ ), and to zero if at least one of  $x_i$  is much

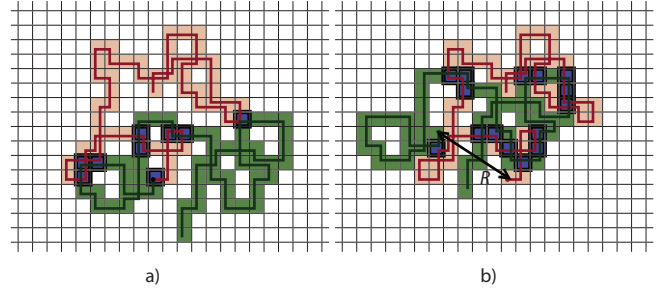


Figure 1: (Color online) A realization of two random walks, starting at the origin (a), and at two points separated by distance  $R$  (b). The sites visited by *both* walks are denoted by dark squares.

larger than 1 (indeed, if the starting positions are separated by a distance much larger than  $t^2$ , the probability of any overlap is exponentially small). In this paper, we show that the scaling function  $\Phi_d(\xi)$  can be calculated exactly in the case of two random walkers starting at a distance  $R$  from each other (see Fig. 1).

We consider two random walks of given lengths  $t_1, t_2$ , their starting points  $x_1, x_2$  being separated by the distance  $R = |x_1 - x_2|$ . We are interested in calculating the average volume  $w_2$  of the domain visited by both random walkers as a function of  $t_1, t_2$  and  $R$ . Following [12] we express this volume in terms of the probability that a given site  $x$  is visited by each of the walkers

$$w_2(x_1, x_2, t_1, t_2) = \int p(x, t_1 | x_1) p(x, t_2 | x_2) d^d x. \quad (2)$$

Here  $p(x, t | x_0)$  is the probability that a random walk starting at  $x_0$  has visited a point  $x$  by time  $t$ , and the integral should be replaced with the sum for walkers in discrete space (lattice).

Now, the probability  $p(x, t | x_0)$  can be expressed in terms of the random walk propagator  $g(x, t | x_0)$  as fol-

lows

$$p(x, t|x_0) = \int_0^t g(x, \tau|x_0) q(t - \tau|x) d\tau, \quad (3)$$

where  $q(t|x)$  is the persistence probability at point  $x$ , i.e. the probability that a walker starting from the point  $x$  does not return to it up to time  $t$ . There is a simple relation between the persistence probability and the probability  $f(t|x)$  of the first return at the point  $x$ :

$$\frac{\partial q(t|x)}{\partial t} = -f(t|x), \quad q(0|x) = 1, \quad (4)$$

from which

$$q(t|x) = 1 - \int_0^t f(t'|x) dt'. \quad (5)$$

Substituting Eq. (3) into Eq. (2) yields

$$\begin{aligned} w_2(x_1, x_2, t_1, t_2) &= \int_0^{t_1} d\tau_1 \int_0^{t_2} d\tau_2 \int q(t_1 - \tau_1|x) \\ &\times q(t_2 - \tau_2|x) g(x, \tau_1|x_1) g(x, \tau_2|x_2) d^d x. \end{aligned} \quad (6)$$

In the most interesting case of time-reversible translationally invariant random walks,  $q(t|x) = q(t)$  is site-independent, and  $g(x, t|x_0) = g(x_0, t|x)$ , so that Eq. (6) can be further simplified into

$$\begin{aligned} w_2(x_1, x_2, t_1, t_2) &= \int_0^{t_1} d\tau_1 \int_0^{t_2} d\tau_2 q(t_1 - \tau_1) q(t_2 - \tau_2) g(x_2, \tau_1 + \tau_2|x_1). \end{aligned} \quad (7)$$

For Brownian random walks in the whole space  $\mathbb{R}^n$ , the propagator is Gaussian (for discrete space, it is asymptotically Gaussian at large  $t$ )

$$g(x_1, t|x_2) = (4\pi Dt)^{-d/2} \exp\left(-\frac{|x_1 - x_2|^2}{4Dt}\right), \quad (8)$$

where the diffusion coefficient  $D = a^2/(2d\delta)$  is related to a microscopic length  $a$  of the order of underlying lattice spacing, and  $\delta$  being the duration of time step. In addition, the persistence was well-studied and its asymptotic behavior depends crucially on the dimensionality of space (in particular, on whether the walk is recurrent or transient) [3, 15]:

$$q(t) \sim \begin{cases} t^{-d/2} & (d < 2), \\ (\ln t)^{-1} & (d = 2), \\ \text{const} + O(t^{-(d-2)/2}) & (d > 2), \end{cases} \quad (9)$$

where the proportionality constants depend in general on both  $d$  and the structure of the underlying lattice. In what follows, we substitute Eqs. (8, 9) into Eq. (7) for

the particular case  $t = t_1 = t_2$  in order to calculate the scaling function

$$\Phi_d(\xi) \equiv \frac{w_2(0, R, t, t)}{w_2(0, 0, t, t)}, \quad \xi \equiv \frac{R}{\sqrt{4dDt}} = \frac{R/a}{\sqrt{2t/\delta}}. \quad (10)$$

It is convenient to consider separately the two cases (i)  $d < 2$ , and (ii)  $d \geq 2$ .

(i) *Dimensionality less than 2.* Substituting Eqs. (7, 8, 9) into Eq. (10) yields the scaling function in the following dimensionless form

$$\begin{aligned} \Phi_d(\xi) &\simeq \frac{I_{<}(\xi, d)}{I_{<}(0, d)}, \\ I_{<}(\xi, d) &= \int_0^1 \int_0^1 \frac{dz_1}{(1 - z_1)^{d/2}} \frac{dz_2}{(1 - z_2)^{d/2}} \frac{\exp(-\frac{d\xi^2}{z_1 + z_2})}{(z_1 + z_2)^{d/2}}. \end{aligned} \quad (11)$$

This integral can be further simplified by changing variables as  $u = (z_1 + z_2)$ ,  $v = (z_1 - z_2)/2$ :

$$\begin{aligned} I_{<}(\xi, d) &= 2 \int_0^1 du \frac{(1 - u/2)^{1-d}}{u^{d/2}} \exp(-d\xi^2/u) \\ &\times \int_0^{u/(2-u)} \frac{dv}{(1 - v^2)^{d/2}} \\ &+ \frac{\sqrt{\pi} \Gamma(1 - d/2)}{\Gamma(3/2 - d/2)} \\ &\times \int_1^2 du u^{-d/2} (1 - u/2)^{1-d} \exp(-d\xi^2/u). \end{aligned} \quad (12)$$

In particular, for  $d = 1$  one can further simplify this formula into

$$\begin{aligned} I_{<}(\xi, 1) &= 2 \int_0^1 \frac{du}{\sqrt{u}} \exp(-\xi^2/u) \arcsin(u/(2 - u)) \\ &+ 2\pi [\sqrt{2} \exp(-\xi^2/2) - \exp(-\xi^2)] \\ &+ 2\pi\sqrt{\pi} \xi [\text{erf}(\xi/\sqrt{2}) - \text{erf}(\xi)], \end{aligned} \quad (13)$$

where

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-y^2) dy$$

is the error function.

Expanding the above expression into a series in the vicinity of  $\xi = 0$  leads to

$$\Phi_1(\xi) \simeq 1 - \xi^2 \frac{1}{2(\sqrt{2} - 1)} + O(\xi^4). \quad (14)$$

In turn, for large  $\xi$ , the error function converges to one exponentially fast, thus the whole expression in Eq. (13) vanishes exponentially fast.

(ii) *Dimensionality greater than or equal to two.* For  $d \geq 2$  the scaling function gets an even simpler form. Indeed, one easily sees substituting Eqs. (7, 8, 9) into Eq. (10) that in the first approximation the input from

the persistence cancels out and the scaling function reads simply

$$\Phi_d(\xi) \simeq \frac{I_{>}(\xi, d)}{I_{>}(0, d)}, \quad (15)$$

$$I_{>}(\xi, d) = \int_0^1 \int_0^1 dz_1 dz_2 \frac{\exp(-\frac{d\xi^2}{z_1+z_2})}{(z_1+z_2)^{d/2}}.$$

However, for  $d \geq 4$  the integral  $I_{>}(0, d)$  diverges which means (see [12]) that the overlap in this case is controlled by the behavior at small  $t$ , and scaling function does not exist. For  $d = 2, 3$  the integrals in Eq. (15) can be computed exactly by substitution  $u = z_1 + z_2$ ,  $v = (z_1 - z_2)/2$ :

$$I_{>}(\xi, 2) = 2[\exp(-2x^2) - \exp(-x^2)] + 2(1 + 2x^2)\text{Ei}(-2x^2) - 2(x^2 + 1)\text{Ei}(-x^2), \quad (16)$$

where

$$\text{Ei}(x) = -\int_{-x}^{\infty} \exp(-y)/y \, dy$$

is the exponential integral function; and

$$I_{>}(\xi, 3) = 4\exp(-3\xi^2) - 2\sqrt{2}\exp(-3\xi^2/2) - 2\sqrt{3\pi} \xi + \sqrt{3\pi} \left(4\xi + \frac{2}{3\xi}\right) \text{erf}(\sqrt{3}\xi) - \sqrt{3\pi} \left(2\xi + \frac{2}{3\xi}\right) \text{erf}(\sqrt{3/2}\xi). \quad (17)$$

For small  $\xi$ , the scaling functions behave as

$$\Phi_2(\xi) \simeq 1 + \frac{2 \ln \xi}{\ln 2} \xi^2 + \frac{2 \ln 2 + \gamma - 2}{\ln 2} \xi^2 - 3\xi^4/2 + O(\xi^6), \quad (18)$$

where  $\gamma \approx 0.577216$  is the Euler's gamma constant, and

$$\Phi_3(\xi) \simeq 1 - \xi \frac{\sqrt{3\pi}}{2\sqrt{2}(\sqrt{2}-1)} + \xi^2 \frac{3+\sqrt{2}}{2} + O(\xi^4). \quad (19)$$

Note that contrary to Eq. (14), the scaling function in  $d = 2, 3$  has a singularity in the vicinity of  $R = 0$ : indeed, the first corrections are proportional to  $R^2 \ln R$  and  $R = \sqrt{R_x^2 + R_y^2 + R_z^2}$ , respectively. For  $\xi \gg 1$ , both  $I_{>}(\xi, 2)$  and  $I_{>}(\xi, 3)$  vanish exponentially fast as expected.

Figure 2 shows the scaling functions  $\Phi_d(\xi)$  for  $d = 1, 2, 3$  and their asymptotic behaviors. To check the results presented above, we simulated random walks on a (hyper)cubic lattice in  $d = 1, 2, 3, 4$  for initial distances equal to  $R = 5, 10, 20, 50$ . The results are presented in Fig. 3 (note the logarithmic scale of the horizontal axis). The theoretical results given by Eqs. (13), (16), and (17) are shown by thick black lines. Note the absence of any scaling collapse of the curves for  $d = 4$ .

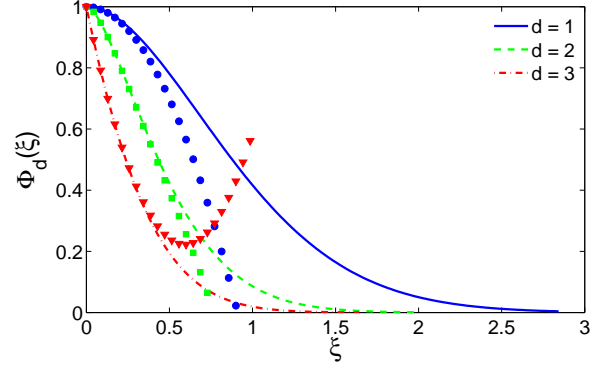


Figure 2: (Color online) Scaling functions  $\Phi_d(\xi)$  for  $d = 1, 2, 3$  (lines) and their asymptotic behaviors in Eqs. (14, 18, 19) for  $0 < \xi < 1$  (symbols).

In summary, we have shown that the average volume visited by *both* independent random walkers starting initially at some given distance  $R$  from each others behaves in a strikingly universal way as a function of the scaling variable  $\xi = R/\sqrt{2t}$ . It is instructive to consider the results in terms of three different phases of the overlap scaling discussed in [12]. In the low-dimensionality phase where overlap for  $\xi = 0$  scales as  $w_2(0, 0, t, t) \sim t^{d/2}$ , corrections due to a non-zero initial distance are analytical functions of that distance

$$\frac{w_2(0, R, t, t)}{w_2(0, 0, t, t)} = 1 - a(d)R^2/t + O(R^4/t^2), \quad (20)$$

with some  $d$ -dependent correction constant  $a(d)$ , e.g.,  $a(1) = \frac{1}{4(\sqrt{2}-1)}$  according to Eq. (14). In the medium-dimensionality phase  $2 < d < 4$ , where  $w_2(0, 0, t, t) \sim t^{(4-d)/2}$ , the introduction of a non-zero initial distance gives rise to a correction which is singular at  $R = 0$

$$\frac{w_2(0, R, t, t)}{w_2(0, 0, t, t)} = 1 - a(d)(R^2/t)^{(4-d)/2} + O(R^2/t), \quad (2 < d < 4), \quad (21)$$

$$\frac{w_2(0, R, t, t)}{w_2(0, 0, t, t)} = 1 + \frac{\ln(R^2/t)}{\ln 2} R^2/t + O(R^2/t), \quad (d = 2).$$

Finally, in the large-dimensionality phase with  $d \geq 4$  where the overlap is mostly controlled by the small  $t$  behavior of the walks, no scaling function exists at all.

It is also instructive to consider the difference  $w_2(0, 0, t, t) - w_2(0, R, t, t)$ , i.e. the average “deficiency” of the overlap function due to the walks starting at distance  $R$  from each other. For  $d < 2$  this difference converges to zero at large  $t$  as  $R^2 t^{(d-2)/2}$ , while for  $2 < d < 4$  it converges to a finite limit which scales as  $R^{4-d}$ .

One other interesting qualitative result illustrating the behavior in the  $2 < d < 4$  region concerns the fraction of the sites visited by the first walk which are also visited

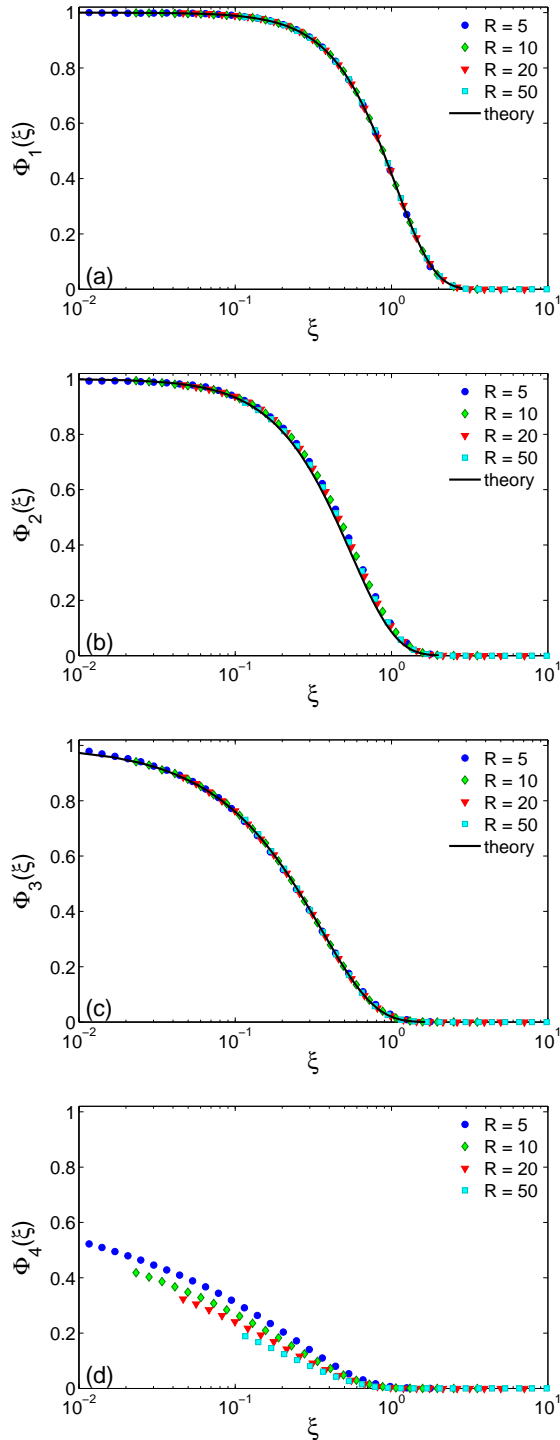


Figure 3: (Color online). Numerical results for the renormalized overlap functions  $\Phi_d(\xi) = w_2(0, R, t, t)/w_2(0, 0, t, t)$  for  $d = 1$  (a),  $d = 2$  (b),  $d = 3$  (c), and  $d = 4$  (d). Numerical results were obtained by Monte Carlo simulations of  $d$ -dimensional random walks on (hyper)cubic lattices with  $R = 5$  (circles), 10 (diamonds), 20 (triangles), and 50 (stars). Each point is an average over 262 144 realizations of random walks up to  $t = 131\,072$  steps. The theoretical predictions for the scaling curves in  $d = 1, 2, 3$  are shown with solid black lines.

by the second walk, i.e. the ratio

$$f_d(R, t) = \frac{w_2(0, R, t, t)}{w_1(t)} = \frac{w_2(0, R, t, t)}{w_2(0, 0, t, t)} \frac{w_2(0, 0, t, t)}{w_1(t)}. \quad (22)$$

The two ratios on the right hand-side of Eq. (22) are both positive and converge to zero as  $t \rightarrow 0$  or  $t \rightarrow \infty$ . Therefore, for any given  $R$  there exists time (of order  $R^2$ ) at which the relative overlap of two walks is maximal:  $f_d(R, t) = f_d^{\max}(R) \sim R^{2-d}$ .

The results on the average volume of several random walks can be of practical use to estimate, e.g. the interactions and entanglements of Gaussian polymer coils, or the oversampling rate in intermittent search processes where the search for the target is an alternating sequence of random walks and longer jumps (see [16] for examples), or surface-mediated diffusion [17–19].

In order to keep the presentation as transparent as possible, we concentrated here on the simplest possible set-up of two walks of equal length. It is clear that the generalization for walks of different lengths is straightforward, and all asymptotical results for  $t \rightarrow \infty$  hold as soon as the two walk lengths remain comparable in this limit. Generalizations for  $N > 2$  walkers are more cumbersome but also straightforward. It may be also interesting to study the overlaps further in confined geometries (e.g., in a  $d$ -dimensional sphere or torus): in this case the overlap fraction  $f_d(R, t)$  can exhibit a peculiar non-monotonous behavior as a function of  $t$ .

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